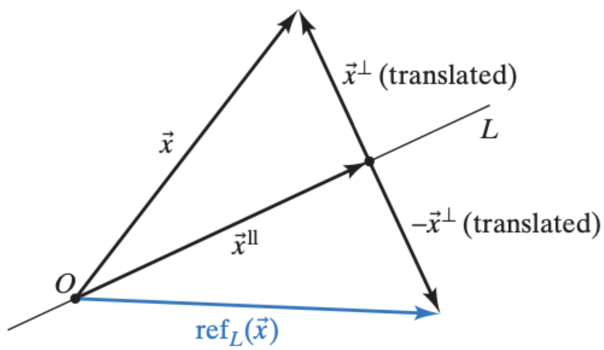
Example Let $\mathbf{x} \in \mathbb{R}^2$, and L be the line spanned by the vector $\mathbf{w}$. Use the following diagram to derive a formula, in terms of the orthogonal projection of $\mathbf{x}$ onto the line L, for the reflection of $\mathbf{x}$ about the line L, $\text{ref}_L(\mathbf{x})$, and prove that it is linear. What can you say about the matrix of this linear transformation?



We derived the following formula for the orthogonal projection of $\mathbf{x}$ onto the line L in class:

$$\text{proj}_L(\mathbf{x}) = \left(\frac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\right) \mathbf{w}$$

Here, $\mathbf{w}$ is a vector parallel to the line L. When $\mathbf{w} = \mathbf{u}$ is a unit vector, the formula becomes

$$\text{proj}_L(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{u})\, \mathbf{u}$$

Let $\mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, where $u_1^2 + u_2^2 = 1$. Then the matrix of the linear transformation $\text{proj}_L(\cdot)$ is given by

$$P = \frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix} = \begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}$$

Let's derive a formula for the reflection of $\mathbf{x}$ about the line L: $\text{ref}_L(\mathbf{x})$

By the diagram, we have

$$\text{ref}_L(\mathbf{x}) = \mathbf{x}^{\parallel} - \mathbf{x}^{\perp}$$

and

$$\mathbf{x} = \text{proj}_L(\mathbf{x}) + \mathbf{x}^{\perp}.$$

By adding two equations, we obtain

$$\text{ref}_L(\mathbf{x}) = 2\text{proj}_L(\mathbf{x}) - \mathbf{x}.$$

A straightforward computation shows that $\text{ref}_L(\cdot)$ is a linear transformation. What is the matrix of the reflection transformation?

$$\text{ref}_L(\mathbf{x}) = 2\text{proj}_L(\mathbf{x}) - \mathbf{x} = (2P - I)\mathbf{x} = \begin{bmatrix} 2u_1^2 - 1 & u_1 u_2 \\ u_1 u_2 & 2u_2^2 - 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Recall that $u_1^2 + u_2^2 = 1$. It is easy to check that $2u_2^2 - 1 = -(2u_1^2 - 1)$ and $(2u_1^2 - 1)^2 + (u_1 u_2)^2 = 1$. Thus, the matrix of the reflection transformation is of the form

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix}.$$

Conversely, any matrix of this form represents a reflection about a line.